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The Fourth-order Difference Equation Satisfied by the Associated Orthogonal Polynomials of the Δ-Laguerre–Hahn Class

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Starting from the D_ω-Riccati difference equation satisfied by the Stieltjes function of a linear functional, we work out an algorithm which enables us to write the general fourth-order difference equation satisfied by the associated of any integer order of orthogonal polynomials of the Δ-Laguerre–Hahn class. Moreover, in classical situations (Meixner, Charlier, Krawtchouk and Hahn), we give these difference equations explicitly; and from the Hahn difference equation, by limit processes we recover the difference equations satisfied by the associated of the classical discrete orthogonal polynomials and the differential equations satisfied by the associated of the classical continuous orthogonal polynomials.

1. Introduction

The associated polynomials of any order of a given Laguerre–Hahn orthogonal sequence belong to the Laguerre–Hahn class (see Magnus, 1984) and therefore satisfy a fourth-order linear differential equation. Belmehdi and Ronveaux (1991) in the general situation have given the fourth-order difference equation satisfied by the associated (of any integer order) of the orthogonal polynomials of the Laguerre–Hahn class, extending works by Wimp (1987).

In this work, using the properties of the linked Stieltjes functions of associated orthogonal polynomials systematically, we prove that associated polynomials of any order of a given Δ-Laguerre–Hahn orthogonal sequence belong to the Δ-Laguerre–Hahn class and we give the general fourth-order difference equation satisfied by the associated of any integer order r. We derive explicitly these difference equations for the associated of order r of the classical orthogonal polynomials (Meixner, Charlier, Krawtchouk, Hahn and Hahn–Eberlein). In the first two cases these results agree with those given by using the hypergeometric representation of the Meixner polynomials (see Letessier *et al.*, 1996). When we set $r = 1$ in the last two cases, we recover other known results (see Ronveaux *et al.*, 1998). From the Hahn case and by limit processes (see Nikiforov *et al.*, 1991; Ronveaux, 1993), we recover both, the fourth-order difference equation for the associated of

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any order of all classical discrete, and the fourth-order differential equation for the associated of any order of all classical continuous cases (see Zarzo *et al.*, 1993), in particular for the Meixner case which was already treated in Letessier *et al.* (1996).

The operator D_ω is defined in Salto (1995)

$$D_\omega f(x) = \frac{f(x+\omega) - f(x)}{\omega}, \qquad (D_1 \equiv \Delta, D_{-1} \equiv \nabla). \tag{1.1}$$

2. D_ω-Riccati Difference Equation for the Stieltjes Function and Some Consequences

2.1. BACKGROUND MATERIAL

Let $\mathcal{L}$ be a regular linear form in the dual of the vector space of all polynomials of one variable (see Chihara, 1978).

By "regular" (see Belmehdi, 1990a) we mean that there exists a sequence of monic polynomials $(P_n)_{n\geq 0}$, called orthogonal with respect to $\mathcal{L}$, satisfying

$$\begin{cases} \text{degree of} P_n = n, & n \geq 0, \\ \langle \mathcal{L}, P_n P_m \rangle = 0 & n \neq m, \\ \langle \mathcal{L}, P_n P_n \rangle \neq 0 & n \geq 0, \end{cases} \tag{2.1}$$

where $\langle \mathcal{L}, P \rangle$ denotes the value of the linear form $\mathcal{L}$ applied to the polynomial P; $\gamma_0 \equiv \langle \mathcal{L}, 1 \rangle$ is the first moment of $\mathcal{L}$.

$(P_n)_n$ satisfy the three-term recurrence relation

$$\begin{cases} P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), & n \geq 1, \\ P_0(x) = 1, P_1(x) = x - \beta_0, \end{cases} \tag{2.2}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0$.

The associated orthogonal polynomial of order r is defined by:

$$\gamma_{r-1} P_n^{(r)}(x) = \left\langle \mathcal{L}^{(r-1)}, \frac{P_{n+1}^{(r-1)}(x) - P_{n+1}^{(r-1)}(t)}{x - t} \right\rangle, \qquad n \geq 0, \qquad r \geq 1, \tag{2.3}$$

assuming that $P_n^{(0)} \equiv P_n$ and $\mathcal{L}^{(0)} \equiv \mathcal{L}$, where $\mathcal{L}^{(r-1)}$ is the regular linear functional with respect to which $(P_n^{(r-1)})_n$ is orthogonal; and it is understood that $\mathcal{L}^{(r-1)}$ acts on the variable t.

Using the three-term recurrence relations satisfied by $(P_n)_n$ and by induction on r, we obtain (see Chihara, 1978)

$$\begin{cases} P_{n+1}^{(r)}(x) = (x - \beta_{n+r}) P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x), & n \geq 1, \\ P_0^{(r)}(x) = 1, P_1^{(r)}(x) = x - \beta_r, \qquad r \geq 0. \end{cases} \tag{2.4}$$

DEFINITION 2.1. The linear form $\mathcal{L}$ and the corresponding orthogonal polynomials are said to be D_ω-semi-classical if $\mathcal{L}$ is regular and there exist two polynomials ψ of degree at least one, and ϕ such that

$$D_\omega(\phi \mathcal{L}) = \psi \mathcal{L}, \tag{2.5}$$

where

$$\begin{cases} \langle \psi \mathcal{L}, P \rangle = \langle \mathcal{L}, \psi P \rangle \\ \langle D_\omega \mathcal{L}, P \rangle = -\langle \mathcal{L}, D_{-\omega} P \rangle. \end{cases} \tag{2.6}$$

Moreover, if ϕ is a polynomial of degree at most two and ψ a polynomial of degree one, then the linear form and the polynomials are called D_ω-classical.

REMARK 2.1. The advantage of the above definition as compared with the original one (see Salto, 1995)

$$\begin{cases} \langle \psi\mathcal{L}, P\rangle = \langle \mathcal{L}, \psi P\rangle \\ \langle D_\omega\mathcal{L}, P\rangle = -\langle \mathcal{L}, D_\omega P\rangle, \end{cases} \tag{2.7}$$

is given in the following comments.

If (P_n) is a classical family of orthogonal polynomials with respect to the regular linear form $\mathcal{L}$, relations (2.6) link equation (2.5) to the second-order difference equation

$$\phi\Delta\nabla P_n + \psi\Delta P_n + \lambda_n P_n = 0, \qquad n \geq 0, \tag{2.8}$$

where λ_n is given by $2\lambda_n = -n[(n-1)\phi'' + 2\psi']$ (see Nikiforov *et al.*, 1991).

In the other direction, relations (2.5) and (2.7) generate the following difference equation (see Salto, 1995) for the same family P_n

$$\phi\Delta\nabla P_n + \psi\nabla P_n + \lambda_n P_n = 0, \qquad n \geq 0, \tag{2.9}$$

an equation which by the relation $\Delta\nabla = \Delta - \nabla$ is equivalent to

$$(\phi - \psi)\Delta\nabla P_n + \psi\Delta P_n + \lambda_n P_n = 0, \qquad n \geq 0. \tag{2.10}$$

2.2. Stieltjes function

The formal Stieltjes function of $\mathcal{L}^{(r)}$ is defined by:

$$S_r(x) = -\sum_{k\geq 0} \frac{\langle \mathcal{L}^{(r)}, x^k\rangle}{x^{k+1}}, \qquad r \geq 0. \tag{2.11}$$

It is a well-known result (see Sherman, 1933; Maroni, 1986) that:

$$S_r(x) = -\frac{\gamma_r}{x - \beta_r + S_{r+1}(x)}, \qquad r \geq 0. \tag{2.12}$$

DEFINITION 2.2. The formal Stieltjes function S_r of the linear form $\mathcal{L}^{(r)}$ satisfies a D_ω-Riccati difference equation if S_r satisfies an equation of type

$$\begin{aligned} \phi(x+\omega)D_\omega S_r(x) &= G_r(x,\omega)S_r(x)\mathcal{T}_\omega S_r(x) + E_r(x,\omega)S_r(x) \\ &\quad + F_r(x,\omega)\mathcal{T}_\omega S_r(x) + H_r(x,\omega), \qquad r \geq 0, \end{aligned} \tag{2.13}$$

where ϕ, E_r, F_r, G_r and H_r are polynomials in the variable x and depending on ω. When $G_r = 0$, the D_ω-Riccati difference equation is called an affine D_ω-Riccati difference equation.

The operator $\mathcal{T}_\omega$ is defined by:

$$\mathcal{T}_\omega f(x) = f(x+\omega), \tag{2.14}$$

and the following properties are easily checked:

$$D_{-\omega}D_\omega = D_\omega D_{-\omega}, D_\omega\mathcal{T}_\omega = \mathcal{T}_\omega D_\omega, D_{-\omega}\mathcal{T}_\omega = \mathcal{T}_\omega D_{-\omega} = D_\omega. \tag{2.15}$$

D_ω and $\mathcal{T}_\omega$ are linked by

$$D_\omega = \frac{\mathcal{T}_\omega - \mathcal{I}_d}{\omega}, \qquad (\mathcal{T}_1 \equiv \mathcal{T}), \tag{2.16}$$

where $\mathcal{I}_d$ is the identity operator and $\mathcal{T}$ the usual shift operator.

Definition 2.3. Let $(P_n)_n$ be a sequence of monic orthogonal polynomials with respect to the linear regular form $\mathcal{L}$.

The family $(P_n)_n$ belongs to the D_ω-Laguerre–Hahn class (resp. the affine D_ω-Laguerre–Hahn class) if the Stieltjes function S_0 of the linear form $\mathcal{L}$ satisfies a D_ω-Riccati difference equation (resp. an affine D_ω-Riccati difference equation) (see equation (2.13)).

Using the above definition and the link between equation (2.5) and the Stieltjes function of $\mathcal{L}$ (see Guerfi, 1988), we remark that the affine D_ω-Laguerre–Hahn class and the set of D_ω-semi-classical orthogonal polynomials are the same.

Proposition 2.1. *If S_r satisfies the D_ω-Riccati difference equation*

$$\begin{aligned}\phi(x+\omega)D_\omega S_r(x) = {} & G_r(x,\omega)S_r(x)\mathcal{T}_\omega S_r(x) + E_r(x,\omega)S_r(x) \\ & + F_r(x,\omega)\mathcal{T}_\omega S_r(x) + H_r(x,\omega), \qquad r \geq 0,\end{aligned} \tag{2.17}$$

where ϕ, E_r, F_r, G_r and H_r are polynomials, then the same property holds for S_{r+1}

$$\begin{aligned}\phi(x+\omega)D_\omega S_{r+1}(x) = {} & G_{r+1}(x,\omega)S_{r+1}(x)\mathcal{T}_\omega S_{r+1}(x) + E_{r+1}(x,\omega)S_{r+1}(x) \\ & + F_{r+1}(x,\omega)\mathcal{T}_\omega S_{r+1}(x) + H_{r+1}(x,\omega),\end{aligned} \tag{2.18}$$

with

$$\begin{aligned}G_{r+1} &= \frac{H_r}{\gamma_r}, \\ E_{r+1} &= (x+\omega-\beta_r)\frac{H_r}{\gamma_r} - F_r, \\ F_{r+1} &= (x-\beta_r)\frac{H_r}{\gamma_r} - E_r,\end{aligned} \tag{2.19}$$

$$\begin{aligned}H_{r+1} = {} & -\phi(x+\omega) + \gamma_r G_r - (x-\beta_r)(E_r+F_r) - \omega E_r \\ & + (x-\beta_r)(x-\beta_r+\omega)\frac{H_r}{\gamma_r}.\end{aligned} \tag{2.20}$$

Proof. Application of the D_ω quotient rule

$$D_\omega\left(\frac{f}{g}\right)(x) = \frac{g(x+\omega)D_\omega f(x) - f(x+\omega)D_\omega g(x)}{g(x)g(x+\omega)}, \tag{2.21}$$

to (2.12) gives

$$D_\omega S_r(x) = \frac{\gamma_r[1 + D_\omega S_{r+1}(x)]}{(x+\omega-\beta_r+\mathcal{T}_\omega S_{r+1}(x))(x-\beta_r+S_{r+1}(x))}. \tag{2.22}$$

Using equations (2.12), (2.17) and (2.22), we obtain the D_ω-Riccati difference equation for S_{r+1}. Identification of this difference equation with equation (2.18) achieves the proof. □

Remark 2.2. (i) From relations (2.19) and (2.20), we obtain

$$E_{r+1} - F_{r+1} - E_r + F_r = \omega \frac{H_r}{\gamma_r}, \tag{2.23}$$

$$E_{r+1} + F_{r+1} + E_r + F_r = 2\frac{H_r}{\gamma_r}(x - \beta_r) + \omega \frac{H_r}{\gamma_r}, \tag{2.24}$$

$$\begin{aligned}\frac{(E_{r+1} + F_{r+1})^2 - (E_r + F_r)^2}{4} &= \phi(x+\omega)\frac{H_r}{\gamma_r} - \frac{H_{r-1}H_r}{\gamma_{r-1}} + \frac{H_r H_{r+1}}{\gamma_r} \\ &\quad + (E_r - F_r)\frac{\omega}{2}\frac{H_r}{\gamma_r} + \frac{\omega^2}{4}(\frac{H_r}{\gamma_r})^2.\end{aligned}$$

(ii) Knowing polynomials ϕ, H_0, E_0, F_0, β_n and γ_n, $n \geq 0$, we can compute the coefficients E_i, F_i and H_i for all $i \geq 1$ using equations (2.19) and (2.20).

(iii) Using the above proposition, one shows by induction on r that the associated of order r of the polynomials belonging to the D_ω-Laguerre–Hahn class belong to the D_ω-Laguerre–Hahn class.

Proposition 2.2. (Guerfi, 1988) *If E_r, F_r, G_r and H_r are the coefficients of the D_ω-Riccati difference equation satisfied by the Stieltjes function S_r of the rth associated $P_n^{(r)}$ of the orthogonal family P_n, then we have*

$$\deg(H_r) \leq m-1, \deg(E_r) \leq m, \qquad \textit{and} \qquad \deg(F_r) \leq m, \qquad r \geq 0, \tag{2.25}$$

where m is given by $m = \max\{\deg(E_0), \deg(F_0), \deg(H_0)+1\}$, and $\deg(E_r)$ denotes the degree of the polynomial E_r with respect to the variable x.

Proof. For $r = 0$, the relation (2.25) holds by hypothesis.

Let us suppose that the relation (2.25) holds up to a fixed r. Using (2.19), we obtain

$$\begin{aligned}\deg(E_{r+1}) &= \deg((x+\omega-\beta_r)\frac{H_r}{\gamma_r} - F_r) \\ &\leq m,\end{aligned} \tag{2.26}$$

by the above hypothesis.

Likewise, using (2.19) we have $\deg(F_{r+1}) \leq m$.

Use of (2.19) and the fact that the last two inequalities of (2.25) hold for any integer r, gives

$$\deg(H_{r+1}) + 1 = \deg(F_{r+1} + E_{r+2}) \leq m. \ \square$$

Proposition 2.3. (Guerfi, 1988) *If $\mathcal{L}$ is a D_ω-semi-classical functional, satisfying $D_\omega(\phi\mathcal{L}) = \psi\mathcal{L}$, then the coefficients E_r, F_r, G_r and H_r of the D_ω-Riccati difference equation satisfied by the Stieltjes function S_r of the rth associated $P_n^{(r)}$ of the orthogonal family P_n satisfy*

$$\begin{aligned}\deg(H_r) &\leq \max\{\deg(\psi)-1, \deg(\phi)-2\}, \qquad r \geq 0, \\ \deg(E_r) &\leq \max\{\deg(\psi), \deg(\phi)-1\}, \qquad r \geq 0, \\ \deg(F_r) &\leq \max\{\deg(\psi), \deg(\phi)-1\}, \qquad r \geq 0,\end{aligned} \tag{2.27}$$

where ϕ is any polynomial, and ψ is a polynomial of degree at least one.

Proof. Let $\mathcal{L}$ be a D_ω-semi-classical functional, satisfying $D_\omega(\phi\mathcal{L}) = \psi\mathcal{L}$. It is a well-known result (see Guerfi, 1988) that the Stieltjes function S_0 of the functional $\mathcal{L}$ satisfies a D_ω -Riccati difference equation

$$\begin{aligned}\phi(x+\omega)D_\omega S_0(x) &= E_0(x,\omega)S_0(x) + F_0(x,\omega)\mathcal{T}_\omega S_0(x) \\ &\quad + G_0(x,\omega)S_0(x)\mathcal{T}_\omega S_0(x) + H_0(x,\omega),\end{aligned} \tag{2.28}$$

with

$$\begin{aligned}H_0(x,\omega) &= \mathcal{L}\theta_0\psi(x) - (D_\omega\mathcal{L})\theta_0\phi(x+\omega) - \mathcal{L}\theta_0 D_\omega\phi(x),\\ E_0(x,\omega) &= \psi(x) - D_\omega\phi(x),\\ F_0(x,\omega) &= G_0(x,\omega) = 0,\end{aligned} \tag{2.29}$$

where the polynomials $\theta_0 P$ and $\mathcal{L}P$ are defined by (see Guerfi, 1988; Salto, 1995)

$$\theta_0 P(x) = \frac{P(x)-P(0)}{x} = \sum_{j=1}^{n} a_j x^{j-1}, \qquad \mathcal{L}P(x) = \sum_{j=0}^{n}\sum_{k=j}^{n} a_k\langle\mathcal{L}, x^{k-j}\rangle x^j, \tag{2.30}$$

with $P(x) = \sum_{j=0}^{n} a_j x^j$.

From (2.29), we deduce that

$$\deg(E_0) \le \max\{\deg(\psi), \deg(\phi)-1\}, \text{ and } \deg(F_0) \le \max\{\deg(\psi), \deg(\phi)-1\}. \tag{2.31}$$

Using (2.29) and (2.30) we obtain

$$\begin{aligned}\deg(\mathcal{L}\theta_0\psi(x)) &\le \deg(\psi) - 1,\\ \deg(\mathcal{L}\theta_0 D_\omega\phi(x)) &\le \deg(\phi) - 2.\end{aligned} \tag{2.32}$$

If we suppose that $\phi(x+\omega) = \sum_{j=0}^{t} \phi_j x^j$, then using (2.30) we obtain

$$\theta_0\phi(x+\omega) = \sum_{j=0}^{t-1}\phi_{j+1}x^j, (D_\omega\mathcal{L})\theta_0\phi(x+\omega) = \sum_{j=0}^{t-1}\tilde{\phi}_{j+1}x^j \tag{2.33}$$

with

$$\tilde{\phi}_{j+1} = \sum_{k=j}^{t-1}\phi_{k+1}\langle D_\omega\mathcal{L}, x^{k-j}\rangle.$$

It turns out that

$$\tilde{\phi}_t = \phi_t\langle D_\omega\mathcal{L}, 1\rangle = -\phi_t\langle\mathcal{L}, D_{-\omega}1\rangle = 0. \tag{2.34}$$

We now deduce from (2.33) and (2.34) that

$$\deg((D_\omega\mathcal{L})\theta_0\phi(x+\omega)) \le \deg(\phi) - 2, \tag{2.35}$$

and then from (2.32) and (2.35) that

$$\deg(H_0) \le \max\{\deg(\psi)-1, \deg(\phi)-2\}. \tag{2.36}$$

Use of (2.31), (2.36) and the Proposition 2.2 achieve the proof.□

3. Fourth-order Difference Equations

The following lemmas and propositions lead us to the fourth-order difference equation satisfied by the associated of order r $P_n^{(r)}$, of the polynomial P_n, belonging to the Δ-Laguerre–Hahn class.

The key to the relation between S_r and the associated orthogonal polynomials $P_n^{(r)}$ is given by the basic identity (see Dzoumba, 1985)

$$S_r = -\gamma_r \frac{P_n^{(r+1)} + S_{n+r+1}P_{n-1}^{(r+1)}}{P_{n+1}^{(r)} + S_{n+r+1}P_n^{(r)}}. \tag{3.1}$$

The first D_ω derivative of equation (3.1) gives

$$\begin{aligned}
&(P_{n+1}^{(r)} + S_{n+r+1}P_n^{(r)})(\mathcal{T}_\omega P_{n+1}^{(r)} + \mathcal{T}_\omega S_{n+r+1}\mathcal{T}_\omega P_n^{(r)})\frac{D_\omega S_r}{\gamma_r}\\
&\quad = (\mathcal{T}_\omega P_n^{(r+1)} D_\omega P_n^{(r)} - \mathcal{T}_\omega P_{n+1}^{(r)} D_\omega P_{n-1}^{(r+1)})S_{n+r+1}\\
&\quad + (\mathcal{T}_\omega P_{n-1}^{(r+1)} D_\omega P_{n+1}^{(r)} - \mathcal{T}_\omega P_n^{(r)} D_\omega P_n^{(r+1)})\mathcal{T}_\omega S_{n+r+1}\\
&\quad - (\mathcal{T}_\omega P_n^{(r)} D_\omega P_{n-1}^{(r+1)} - \mathcal{T}_\omega P_{n-1}^{(r+1)} D_\omega P_n^{(r)})S_{n+r+1}\mathcal{T}_\omega S_{n+r+1}\\
&\quad - \mathcal{T}_\omega P_{n+1}^{(r)} D_\omega P_n^{(r+1)} + \mathcal{T}_\omega P_n^{(r+1)} D_\omega P_{n+1}^{(r)}\\
&\quad + (\mathcal{T}_\omega P_n^{(r+1)}\mathcal{T}_\omega P_n^{(r)} - \mathcal{T}_\omega P_{n-1}^{(r+1)}\mathcal{T}_\omega P_{n+1}^{(r)})D_\omega S_{n+r+1}.
\end{aligned} \tag{3.2}$$

Substituting (3.1) and (3.2) in (2.17) gives a D_ω-Riccati difference equation for S_{n+r+1} to be identified with

$$\begin{aligned}
\phi(x+\omega)D_\omega S_{n+r+1}(x) &= E_{n+r+1}(x,\omega)S_{n+r+1}(x) + F_{n+r+1}(x,\omega)\mathcal{T}_\omega S_{n+r+1}(x)\\
&\quad + \frac{H_{n+r}(x,\omega)}{\gamma_{n+r}} S_{n+r+1}(x)\mathcal{T}_\omega S_{n+r+1}(x) + H_{n+r+1}(x,\omega),
\end{aligned}$$

and using the identity (see Magnus, 1984; Belmehdi, 1990b)

$$P_n^{(r)}P_n^{(r+1)} - P_{n-1}^{(r+1)}P_{n+1}^{(r)} = \prod_{k=r+1}^{n+r}\gamma_k \equiv \Pi_{n,r}, \qquad r \ge 0, \qquad n \ge 1, \tag{3.3}$$

we obtain the following lemma.

Lemma 3.1.

$$\begin{aligned}
\Pi_{n,r}E_{n+r+1} &= -\phi(x+\omega)(\mathcal{T}_\omega P_n^{(r+1)} D_\omega P_n^{(r)} - \mathcal{T}_\omega P_{n+1}^{(r)} D_\omega P_{n-1}^{(r+1)})\\
&\quad - E_r P_{n-1}^{(r+1)}\mathcal{T}_\omega P_{n+1}^{(r)} - F_r P_n^{(r)}\mathcal{T}_\omega P_n^{(r+1)}\\
&\quad + \frac{H_r}{\gamma_r}P_n^{(r)}\mathcal{T}_\omega P_{n+1}^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}}P_{n-1}^{(r+1)}\mathcal{T}_\omega P_n^{(r+1)},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\Pi_{n,r}F_{n+r+1} &= \phi(x+\omega)(\mathcal{T}_\omega P_n^{(r)} D_\omega P_n^{(r+1)} - \mathcal{T}_\omega P_{n-1}^{(r+1)} D_\omega P_{n+1}^{(r)})\\
&\quad - E_r P_n^{(r+1)}\mathcal{T}_\omega P_n^{(r)} - F_r P_{n+1}^{(r)}\mathcal{T}_\omega P_{n-1}^{(r+1)}\\
&\quad + \frac{H_r}{\gamma_r}P_{n+1}^{(r)}\mathcal{T}_\omega P_n^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}}P_n^{(r+1)}\mathcal{T}_\omega P_{n-1}^{(r+1)},
\end{aligned} \tag{3.5}$$

$$\Pi_{n,r}H_{n+r+1} = -\phi(x+\omega)(\mathcal{T}_\omega P_n^{(r+1)} D_\omega P_{n+1}^{(r)} - \mathcal{T}_\omega P_{n+1}^{(r)} D_\omega P_n^{(r+1)})$$

$$
\begin{aligned}
&-E_r P_n^{(r+1)} \mathcal{T}_\omega P_{n+1}^{(r)} - F_r P_{n+1}^{(r)} \mathcal{T}_\omega P_n^{(r+1)} \\
&+\frac{H_r}{\gamma_r} P_{n+1}^{(r)} \mathcal{T}_\omega P_{n+1}^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_n^{(r+1)} \mathcal{T}_\omega P_n^{(r+1)},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\Pi_{n-1,r} H_{n+r} = &-\phi(x+\omega)(\mathcal{T}_\omega P_{n-1}^{(r+1)} D_\omega P_n^{(r)} - \mathcal{T}_\omega P_n^{(r)} D_\omega P_{n-1}^{(r+1)}) \\
&-E_r P_{n-1}^{(r+1)} \mathcal{T}_\omega P_n^{(r)} - F_r P_n^{(r)} \mathcal{T}_\omega P_{n-1}^{(r+1)} \\
&+\frac{H_r}{\gamma_r} P_n^{(r)} \mathcal{T}_\omega P_n^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_{n-1}^{(r+1)} \mathcal{T}_\omega P_{n-1}^{(r+1)}.
\end{aligned} \tag{3.7}
$$

Elimination of $D_\omega P_{n-1}^{(r+1)}$ (and $D_\omega P_n^{(r)}$, respectively) in (3.4), (3.7) and use of (3.3) give

LEMMA 3.2.

$$
\phi(x+\omega) D_\omega P_n^{(r)} = -E_{n+r+1} \mathcal{T}_\omega P_n^{(r)} - F_r P_n^{(r)} + \frac{H_{n+r}}{\gamma_{n+r}} \mathcal{T}_\omega P_{n+1}^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_{n-1}^{(r+1)}, \tag{3.8}
$$

$$
\phi(x+\omega) D_\omega P_{n-1}^{(r+1)} = -E_{n+r+1} \mathcal{T}_\omega P_{n-1}^{(r+1)} + E_r P_{n-1}^{(r+1)} + \frac{H_{n+r}}{\gamma_{n+r}} \mathcal{T}_\omega P_n^{(r+1)} - \frac{H_r}{\gamma_r} P_n^{(r)}. \tag{3.9}
$$

In the same way, elimination of $D_\omega P_n^{(r+1)}$ (and $D_\omega P_{n+1}^{(r)}$, respectively) in (3.5), (3.6) and use of (3.3) give

LEMMA 3.3.

$$
\phi(x+\omega) D_\omega P_{n+1}^{(r)} = F_{n+r+1} \mathcal{T}_\omega P_{n+1}^{(r)} - F_r P_{n+1}^{(r)} - H_{n+r+1} \mathcal{T}_\omega P_n^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_n^{(r+1)}, \tag{3.10}
$$

$$
\phi(x+\omega) D_\omega P_n^{(r+1)} = F_{n+r+1} \mathcal{T}_\omega P_n^{(r+1)} + E_r P_n^{(r+1)} - H_{n+r+1} \mathcal{T}_\omega P_{n-1}^{(r+1)} - \frac{H_r}{\gamma_r} P_{n+1}^{(r)}. \tag{3.11}
$$

REMARK 3.1. As $\lim_{\omega \to 0} D_\omega = \frac{d}{dx}$, we recover the results given for the continuous case (see Wimp, 1987; Belmehdi and Ronveaux, 1991; Ronveaux, 1991) when taking the limit of equations (3.4)–(3.11) as ω tends to zero.

When we replace ω by one and the operator D_ω by $\mathcal{T} - \mathcal{I}_d$ in equations (3.8)–(3.11), we obtain, respectively:

$$
K_1 \mathcal{T} P_n^{(r)} = K_2 P_n^{(r)} + K_3 \mathcal{T} P_{n+1}^{(r)} + K_4 P_{n-1}^{(r+1)}, \tag{3.12}
$$

$$
K_1 \mathcal{T} P_{n-1}^{(r+1)} = K_5 P_{n-1}^{(r+1)} + K_3 \mathcal{T} P_n^{(r+1)} + K_6 P_n^{(r)}, \tag{3.13}
$$

$$
K_7 \mathcal{T} P_{n+1}^{(r)} = K_2 P_{n+1}^{(r)} + K_8 \mathcal{T} P_n^{(r)} + K_4 P_n^{(r+1)}, \tag{3.14}
$$

$$
K_7 \mathcal{T} P_n^{(r+1)} = K_5 P_n^{(r+1)} + K_8 \mathcal{T} P_{n-1}^{(r+1)} + K_6 P_{n+1}^{(r)}, \tag{3.15}
$$

with polynomials $K_i(r,n,x) \equiv K_i, (i = 1, 8)$, given by

$$\begin{aligned}
&K_1(r,n,x) = \phi(x+1) + E_{n+r+1}(x,1), K_2(r,n,x) = \phi(x+1) - F_r(x,1),\\
&K_3(r,n,x) = \frac{H_{n+r}(x,1)}{\gamma_{n+r}}, K_4(r,n,x) = \begin{cases} \gamma_r \frac{H_{r-1}(x,1)}{\gamma_{r-1}}, & \text{if } \gamma \geq 1 \\ \gamma_0 G_0, & \text{if } \gamma = 0 \end{cases}\\
&K_5(r,n,x) = \phi(x+1) + E_r(x,1), K_6(r,n,x) = -\frac{H_r(x,1)}{\gamma_r}, \qquad (3.16)\\
&K_7(r,n,x) = \phi(x+1) - F_{n+r+1}(x,1), K_8(r,n,x) = -\gamma_{n+r+1}\frac{H_{n+r+1}(x,1)}{\gamma_{n+r+1}}.
\end{aligned}$$

Solving equations (3.12) and (3.13) in terms of $\mathcal{T}P_{n+1}^{(r)}$ and $\mathcal{T}P_n^{(r+1)}$, we obtain

$$\mathcal{T}P_{n+1}^{(r)} = \frac{K_1\mathcal{T}P_n^{(r)} - K_2P_n^{(r)} - K_4P_{n-1}^{(r+1)}}{K_3}, \qquad (3.17)$$

$$\mathcal{T}P_n^{(r+1)} = \frac{K_1\mathcal{T}P_{n-1}^{(r+1)} - K_5P_{n-1}^{(r+1)} - K_6P_n^{(r)}}{K_3}. \qquad (3.18)$$

Replacing $\mathcal{T}^2P_{n+1}^{(r)}$, given by the shifted equation (3.14), in the shifted equation (3.12), and taking into account equations (3.17) and (3.18), we obtain

Proposition 3.1.

$$\mathcal{D}_{r,n}[P_n^{(r)}] = \mathcal{N}_{r+1,n-1}[P_{n-1}^{(r+1)}], \qquad (3.19)$$

where

$$\mathcal{D}_{r,n} = a_2\mathcal{T}^2 + a_1\mathcal{T} + a_0\mathcal{I}_d, \mathcal{N}_{r+1,n-1} = \tilde{a}_1\mathcal{T} + \tilde{a}_0\mathcal{I}_d, \qquad (3.20)$$

$$\begin{aligned}
&a_2 = K_{3,0}(K_{1,1}K_{7,1} - K_{3,1}K_{8,1}), \qquad a_1 = -K_{2,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}),\\
&a_0 = K_{3,1}(K_{2,0}K_{2,1} + K_{4,1}K_{6,0}), \qquad (3.21)\\
&\tilde{a}_1 = K_{4,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}), \qquad \tilde{a}_0 = -K_{3,1}(K_{2,1}K_{4,0} + K_{4,1}K_{5,0}),
\end{aligned}$$

with polynomials $K_{i,j}$ given by (3.16) and

$$K_{i,0}(r,n,x) \equiv K_i(r,n,x), \qquad \text{and} \qquad K_{i,j}(r,n,x) \equiv K_i(r,n,x+j).$$

By the same process, using $\mathcal{T}^2P_n^{(r+1)}$given by the shifted equation (3.15), in the shifted equation (3.13), and taking into account equations (3.17) and (3.18), we obtain moreover:

Proposition 3.2.

$$\bar{\mathcal{D}}_{r+1,n-1}[P_{n-1}^{(r+1)}] = \bar{\mathcal{N}}_{r,n}[P_n^{(r)}], \qquad (3.22)$$

where

$$\bar{\mathcal{D}}_{r+1,n-1} = b_2\mathcal{T}^2 + b_1\mathcal{T} + b_0\mathcal{I}_d, \bar{\mathcal{N}}_{r,n} = \tilde{b}_1\mathcal{T} + \tilde{b}_0\mathcal{I}_d, \qquad (3.23)$$

$$\begin{aligned}
&b_2 = K_{3,0}(K_{1,1}K_{7,1} - K_{3,1}K_{8,1}), b_1 = -K_{5,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}),\\
&b_0 = K_{3,1}(K_{5,0}K_{5,1} + K_{4,0}K_{6,1}), \qquad (3.24)\\
&\tilde{b}_1 = K_{6,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}), \tilde{b}_0 = -K_{3,1}(K_{5,1}K_{6,0} + K_{6,1}K_{2,0}).
\end{aligned}$$

Replacing $\mathcal{T}^2 P_{n-1}^{(r+1)}$, given by (3.22), in the shifted equation (3.19), we obtain moreover:

$$c_3\mathcal{T}^3 P_n^{(r)} + c_2\mathcal{T}^2 P_n^{(r)} + c_1\mathcal{T} P_n^{(r)} + c_0 P_n^{(r)} = \tilde{c}_1\mathcal{T} P_{n-1}^{(r+1)} + \tilde{c}_0 P_{n-1}^{(r+1)}. \tag{3.25}$$

By the same process, replacing $\mathcal{T}^2 P_{n-1}^{(r+1)}$given by (3.22) in the shifted equation (3.25), we obtain

$$d_4\mathcal{T}^4 P_n^{(r)} + d_3\mathcal{T}^3 P_n^{(r)} + d_2\mathcal{T}^2 P_n^{(r)} + d_1\mathcal{T} P_n^{(r)} + d_0 P_n^{(r)} = \tilde{d}_1\mathcal{T} P_{n-1}^{(r+1)} + \tilde{d}_0 P_{n-1}^{(r+1)}, \tag{3.26}$$

where the polynomial coefficients $c_i, \tilde{c}_i, d_i$ and $\tilde{d}_i$ are easily computed from the coefficients $a_i, \tilde{a}_i, b_i$ and $\tilde{b}_i$.

Now, use of equations (3.19), (3.25) and (3.26) give the expected fourth-order difference equation satisfied by each $P_n^{(r)}$

$$\begin{vmatrix} a_2\mathcal{T}^2 P_n^{(r)} + a_1\mathcal{T} P_n^{(r)} + a_0 P_n^{(r)} & \tilde{a}_1 & \tilde{a}_0 \\ c_3\mathcal{T}^3 P_n^{(r)} + c_2\mathcal{T}^2 P_n^{(r)} + c_1\mathcal{T} P_n^{(r)} + c_0 P_n^{(r)} & \tilde{c}_1 & \tilde{c}_0 \\ d_4\mathcal{T}^4 P_n^{(r)} + d_3\mathcal{T}^3 P_n^{(r)} + d_2\mathcal{T}^2 P_n^{(r)} + d_1\mathcal{T} P_n^{(r)} + d_0 P_n^{(r)} & \tilde{d}_1 & \tilde{d}_0 \end{vmatrix} = 0, \tag{3.27}$$

which can be written in the form

$$\sum_{j=0}^{4} I_j(r,n,x)\mathcal{T}^j P_n^{(r)}(x) = 0. \tag{3.28}$$

REMARK 3.2. We have used Maple V Release 4 (see Char *et al.*, 1991) to compute and factorize the coefficients $I_j(r,n,x)$ for the Charlier, Krawtchouk and Meixner cases (see the results in Section 4.3). The Hahn and Hahn–Eberlein cases need heavy computation to receive readable results, and we had to use both Maple V release 4 and Reduce 3.6 (see Hearn, 1995) to compute and factorize the coefficients $I_j(r,n,x)$ (see the results in Section 4.3.4). In fact, there is a memory problem with Maple V.4 when factorizing the coefficients $I_j(r,n,x), j = 1,\ldots,3$. Symbolic computation in this work was absolutely inevitable because these heavy computations cannot be done by human efforts.

4. Application

4.1. Coefficients of the Δ-Riccati difference equation for the associated of the classical cases

Here we suppose that the regular functional $\mathcal{L}$ satisfies the Pearson functional equation $\Delta(\sigma\mathcal{L}) = \tau\mathcal{L}$, where σ is a polynomial of degree at most two, and τ is a polynomial of degree one. It follows from Proposition 2.3 that H_r is constant and E_r, F_r are polynomials of degree at most one.

Let us compute first polynomials E_r, F_r and $\frac{H_r}{\gamma_r}$ in terms of σ and τ.

The first Δ-derivative of (2.23), (2.24), the first and second Δ-derivative of (2.20) in which ϕ replaced by σ and ψ by τ, give, respectively:

$$\Delta E_{r+1} - \Delta F_{r+1} - (\Delta E_r - \Delta F_r) = 0, \qquad r \geq 0, \tag{4.1}$$

$$\Delta(E_{r+1} + F_{r+1}) + \Delta(E_r + F_r) = 2\frac{H_r}{\gamma_r}, \qquad r \geq 0, \tag{4.2}$$

$$E_r + F_r = -\Delta\sigma(x+1) - (\Delta E_r + \Delta F_r)(x - \beta_r + 1) \\ - \Delta E_r + 2(x - \beta_r + 1)\frac{H_r}{\gamma_r}, \tag{4.3}$$

$$2\Delta(E_r + F_r) = -\sigma'' + 2\frac{H_r}{\gamma_r} \qquad r \geq 0. \tag{4.4}$$

Use of (2.29), (4.1), (4.2) and (4.4) gives

$$\Delta E_r = \frac{1}{2}(r-2)\sigma'' + \tau', \qquad \Delta F_r = \frac{r}{2}\sigma'',$$
$$\frac{H_r}{\gamma_r} = \frac{1}{2}(2r-1)\sigma'' + \tau', \qquad r \geq 0. \tag{4.5}$$

We compute E_r and F_r using (2.29), (4.3), (4.5), and the equation obtained after iterating (2.23)

$$E_r - F_r = \sum_{k=0}^{r-1} \frac{H_k}{\gamma_k} - \Delta\sigma(x) + \tau(x)$$

and we obtain

$$E_r(x,1) = \tau(x) - \frac{\tau(\beta_r)}{2} + r\frac{\tau'}{2} + (r^2 - r(1+2\beta_r) - 2)\frac{\sigma''}{4} \\ +(r-2)\frac{\sigma'(x)}{2} - \sigma'(0)\frac{r}{2}, \tag{4.6}$$

$$F_r(x,1) = -\frac{\tau(\beta_r)}{2} - r\frac{\tau'}{2} - (r^2 - r(3-2\beta_r))\frac{\sigma''}{4} + (\sigma'(x) - \sigma'(0))\frac{r}{2}, \tag{4.7}$$

where the coefficients β_r and γ_r are given by (see Lesky, 1985; Koepf and Schmersau, 1998)

$$\beta_n = \frac{-\tau(0)(\tau' - \sigma'') - n(\tau' + 2\sigma'(0))(\tau' + (n-1)\frac{\sigma''}{2})}{((n-1)\sigma'' + \tau')(n\sigma'' + \tau')}, \qquad n \geq 0,$$
$$\gamma_n = -\frac{n(\tau' + (n-2)\frac{\sigma''}{2})}{(\tau' + (2n-3)\frac{\sigma''}{2})(\tau' + (2n-1)\frac{\sigma''}{2})}[\sigma(\eta_{n-1}) + \tau(\eta_{n-1})], \qquad n \geq 1, \tag{4.8}$$
$$\eta_n = -\frac{\tau(0) + n\sigma'(0) - n^2\frac{\sigma''}{2}}{\tau' + n\sigma''}, \qquad n \geq 0.$$

The results in this section allow us to write a fourth-order difference equation (3.27) in terms of σ and τ which therefore is valid for all classical discrete families (see Foupouagnigni *et al.*, 1998) in the same spirit as for the classical continuous families (see Zarzo *et al.*, 1993).

4.2. FACTORIZED FORM OF THE FOURTH-ORDER DIFFERENCE EQUATION SATISFIED BY THE FIRST ASSOCIATED

Use of relations (3.19) and (3.22) for $r = 0$ and taking into account equations (3.16), (4.5), (4.6) and (4.7) gives

$$\mathcal{D}_{0,n} = \sigma_2((\sigma_1 + \tau_1)\mathcal{T}^2 - (2\sigma_1 + \tau_1 - \lambda_n)\mathcal{T} + \sigma_1\mathcal{I}_d) \equiv \sigma_2\mathcal{D}^*_{0,n}, \mathcal{N}_{1,n-1} = 0,$$

$$\bar{\mathcal{D}}_{1,n-1} = (\sigma_1+\tau_1)(\sigma_2\mathcal{T}^2 - (2\sigma_1+\tau_1-\lambda_n)\mathcal{T} + (\sigma_0+\tau_0)\mathcal{I}_d) \equiv (\sigma_1+\tau_1)\bar{\mathcal{D}}^*_{1,n-1},$$

$$\bar{\mathcal{N}}_{0,n} = (\frac{\sigma_0''}{2} - \tau')((2\sigma_1+\tau_1-\lambda_n)\mathcal{T} - (2\sigma_1+\tau_1)\mathcal{I}_d), \tag{4.9}$$

where

$$\sigma_i \equiv \sigma(x+i), \qquad \tau_i \equiv \tau(x+i), \qquad \text{and} \qquad \lambda_n = -\frac{n}{2}[(n-1)\sigma'' + 2\tau']. \tag{4.10}$$

Together with relations (3.19), (3.22) and equation (4.9) this yields

$$\mathcal{D}^*_{0,n}[P_n] = 0, \tag{4.11}$$

$$\bar{\mathcal{D}}^*_{1,n-1}[P^{(1)}_{n-1}] = e_1\mathcal{T}P_n + e_0P_n, \tag{4.12}$$

where the coefficients e_1 and e_0 are easily computed from (3.19), (3.22) and (4.9).

It turns out that equation (4.11) is equivalent to the second-order difference equation satisfied by the classical orthogonal polynomials of a discrete variable

$$\sigma\Delta\nabla P_n + \tau\Delta P_n + \lambda_n P_n = 0.$$

Replacing $\mathcal{T}^2P_n$given by (4.11) in the shifted equation (4.12), we obtain

$$\mathcal{T}\bar{\mathcal{D}}^*_{1,n-1}[P^{(1)}_{n-1}] = f_1\mathcal{T}P_n + f_0P_n. \tag{4.13}$$

By the same process, replacing again $\mathcal{T}^2P_n$given by (4.11) in the shifted equation (4.13), we obtain

$$\mathcal{T}^2\bar{\mathcal{D}}^*_{1,n-1}[P^{(1)}_{n-1}] = g_1\mathcal{T}P_n + g_0P_n, \tag{4.14}$$

where the coefficients f_1, f_0g_1 and g_0 are easily computed from (4.9)–(4.12).

Now the use of equations (4.12)–(4.14) gives the fourth-order difference equation satisfied by $P^{(1)}_{n-1}$

$$\begin{vmatrix} \bar{\mathcal{D}}^*_{1,n-1}[P^{(1)}_{n-1}] & e_1 & e_0 \\ \mathcal{T}\bar{\mathcal{D}}^*_{1,n-1}[P^{(1)}_{n-1}] & f_1 & f_0 \\ \mathcal{T}^2\bar{\mathcal{D}}^*_{1,n-1}[P^{(1)}_{n-1}] & g_1 & g_0 \end{vmatrix} = 0, \tag{4.15}$$

which can be written in the factorized form:

Lemma 4.1. *The fourth-order difference equation satisfied by the first associated $P^{(1)}_{n-1}$ of the classical orthogonal polynomial of a discrete variable is given in the factorized form by (see Atakishiyev* et al.*, 1996; Ronveaux* et al.*, 1998)*

$$\bar{\mathcal{D}}^{**}_{1,n-1}\bar{\mathcal{D}}^*_{1,n-1}[P^{(1)}_{n-1}] = 0, \tag{4.16}$$

with

$$\bar{\mathcal{D}}^*_{1,n-1} = \sigma_2\mathcal{T}^2 - (2\sigma_1+\tau_1-\lambda_n)\mathcal{T} + (\sigma_0+\tau_0)\mathcal{I}_d, \tag{4.17}$$

$$\begin{aligned}\bar{\mathcal{D}}^{**}_{1,n-1} = {} & \lambda_n(\sigma_3+\tau_3)(\lambda_n - 2\sigma_1 - \tau_1 - 2\sigma_2 - \tau_2)\mathcal{T}^2 \\ & +\{(2\sigma_3+\tau_3-\lambda_n)(\sigma_2(2\sigma_1+\tau_1) - (2\sigma_1+\tau_1-\lambda_n)(2\sigma_2+\tau_2-\lambda_n)). \\ & +(\sigma_2+\tau_2)(2\sigma_3+\tau_3)(2\sigma_1+\tau_1-\lambda_n)\}\mathcal{T} + \lambda_n\sigma_1(\lambda_n - 2\sigma_2 - \tau_2 - 2\sigma_3 - \tau_3)\mathcal{I}_d.\end{aligned} \tag{4.18}$$

4.3. RESULTS ON GENERAL ASSOCIATED CLASSICAL DISCRETE POLYNOMIALS

The basic operators (3.20) and (3.23) and the coefficients of the fourth-order difference equation (3.28) are written down in each case (for notations see Nikiforov *et al.*, 1991).

CHARLIER CASE $C_n^{\mu}(x),\ \mu > 0$

$$\sigma(x) = x, \qquad \tau(x) = -x + \mu,$$

$$\begin{aligned}
\mathcal{D}_{r,n} &= \mu(2+x)\mathcal{T}^2 - (2+x-r)(x-n-r+1+\mu)\mathcal{T} \\
&\quad -(-3x-2+3r-x^2+2xr-r^2+r\mu)\mathcal{I}_d, \\
\mathcal{N}_{r+1,n-1} &= -r\mu(x-n-r+1+\mu)\mathcal{T} + r\mu(\mu+2+x-r)\mathcal{I}_d,
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{D}}_{r+1,n-1} &= \mu(2+x)\mathcal{T}^2 - \mu(x-n-r+1+\mu)\mathcal{T} + \mu(-r+\mu)\mathcal{I}_d, \\
\bar{\mathcal{N}}_{r,n} &= -(-x+n+r-1-\mu)\mathcal{T} - (\mu+x+1-r)\mathcal{I}_d,
\end{aligned}$$

$$\begin{aligned}
I_0(r,n,x) &= \mu(1+x)(-2+n+2R), \\
I_1(r,n,x) &= 2x\mu + 2R + 4\mu - 2R^3 + nR - 3nR^2 - n^2R, \\
I_2(r,n,x) &= 2x\mu + 2R + 4\mu - 5\mu n - 2x\mu n - 4\mu xR + 4R^3 - 10\mu R - 6R^2 \\
&\quad -4nR + 6nR^2 + 4n^2R - n^2 + n^3, \\
I_3(r,n,x) &= -2x\mu - 4R - 6\mu - 2n - 2R^3 + 6R^2 + 7nR - 3nR^2 - n^2R + 2n^2, \\
I_4(r,n,x) &= \mu(4+x)(n+2R),
\end{aligned}$$

where R is given by $R = r - x - \mu - 2$.

MEIXNER CASE $M_n^{(\nu,\mu)}(x),\ \nu > 0,\ 0 < \mu < 1$

$$\sigma(x) = x; \qquad \tau(x) = (\mu-1)x + \mu\nu,$$

$$\begin{aligned}
\mathcal{D}_{r,n} &= \mu(x+2)(x+1+\nu)(\mu-1)\mathcal{T}^2 \\
&\quad +(-2-x+r)(1+x-r-n+\mu\nu+x\mu+r\mu+n\mu+\mu)(\mu-1)\mathcal{T} \\
&\quad -(-r\mu+r\mu\nu+r^2\mu-3x-r^2+2xr-x^2-2+3r)(\mu-1)\mathcal{I}_d,
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{r+1,n-1} &= r(\nu+r-1)(1+x-r-n+\mu\nu+x\mu+r\mu+n\mu+\mu)\mu\mathcal{T} \\
&\quad -r(\nu+r-1)(\mu\nu+x\mu+x-r+r\mu+2)\mu\mathcal{I}_d,
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{D}}_{r+1,n-1} &= \mu(x+2)(x+1+\nu)\mathcal{T}^2 \\
&\quad -(x+1+\nu+r)(1+x-r-n+\mu\nu+x\mu+r\mu+n\mu+\mu)\mu\mathcal{T} \\
&\quad +(r-r\nu+x\mu+r\mu+r^2\mu+\mu\nu^2+x^2\mu+2x\mu\nu+2r\mu\nu \\
&\quad +2x\mu r+\mu\nu-r^2)\mu\mathcal{I}_d,
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{N}}_{r,n} &= -(\mu-1)(1+x-r-n+\mu\nu+x\mu-r\mu+n\mu+\mu)\mathcal{T} \\
&\quad +(\mu-1)(x\mu+\mu+\mu\nu+r\mu-r+x+1)\mathcal{I}_d,
\end{aligned}$$

$$I_0(r,n,x) = -\mu(-3\mu+M+2R-3)(x+\nu)(x+1),$$

$$\begin{aligned}
I_1(r,n,x) &= -6\mu^2x - 2x^2\mu^2 - 4\mu^2 - 4\mu^2\nu - 2x\mu^2\nu - 3\mu R^2 - 3\mu MR - 6x\mu \\
&\quad -2x^2\mu - 4\mu - 4\mu\nu - 2x\mu\nu - 3MR - 3R^2 + M^2R + 3R^2M + 2R^3,
\end{aligned}$$

$$
\begin{aligned}
I_2(r,n,x) = & -4R^3-6\mu^2-4M^2R-9\mu^2\nu-14\mu^2x-4\mu R-2\mu M-5M-M^3\\
& -6\mu-4x^2\mu^2-4x^2\mu-4x\mu\nu-4x\mu^2\nu-14x\mu-9\mu\nu+2+16Rx\mu+10R\mu\nu\\
& +12\mu R^2+4Rx^2\mu+4Rx\mu\nu+8\mu Mx+12\mu MR+12MR+2\mu^3+4M^2-10\mu^2R\\
& +5M\mu\nu-6R^2M+2x^2\mu M-5\mu^2M+2x\mu\nu M-10R+12R^2+4\mu M^2,
\end{aligned}
$$

$$
\begin{aligned}
I_3(r,n,x) = & \, 2R^3+M^2R+6\mu^2\nu+10\mu^2x+24\mu R+12\mu M+6M+2x^2\mu^2\\
& +2x^2\mu+2x\mu\mu+2x\mu^2\nu+10x\mu+6\mu\nu-4-9\mu R^2-9\mu MR-9NR-4\mu^3\\
& -2M^2+12\mu^2R+3R^2M+6\mu^2M+12R-9R^2-2\mu M^2,
\end{aligned}
$$

$$
I_4(r,n,x) = -(x+4)(x+3+\nu)(-\mu+M+2R-1)\mu,
$$

where $R=r-x-2-\mu(r+x+\nu)$, and $M=(n+1)(1-\mu)$.

Krawtchouk case $k_n^{(p)}(x),\ p>0,\ q>0,\ p+q=1$

$$
\sigma(x)=x, \qquad \tau(x)=\frac{1}{q}((1-q)N-x),
$$

$$
\begin{aligned}
\mathcal{D}_{r,n} = & \,(q-1)(x+2)(-x-1+N)\mathcal{T}^2\\
& +(-2-x+r)(-2q-2xq+qN-N+x+r+1+n)\mathcal{T}\\
& +(-3xq-2q-Nrq+r-r^2-x^2q+rN+2xqr+2rq)\mathcal{I}_d,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}_{r+1,n-1} = & \,(q-1)(N-r+1)(-2q-2xq+qN-N+x+r+1+n)r\mathcal{T}\\
& +(q-1)(N-r+1)(qN-N-2q-2xq+r+x)r\mathcal{I}_d,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathcal{D}}_{r+1,n-1} = & \, q(q-1)(x+2)(-x-1+N)\mathcal{T}^2\\
& +(q-1)(N-x-r-1)(-2q-2xq+qN-N+x+r+1+n)\mathcal{T}\\
& -(q-1)(-2xqN+qN^2-qN+2rq+2xqr+x^2q+xq-Nrq+2rN\\
& -r^2-r-2xr-N^2+N+2Nx-x^2-x)\mathcal{I}_d,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathcal{N}}_{r,n} = & -(-x-r-qN+2q+2xq-n-1+N)\mathcal{T}-(-2xq-2q+qN+r+x-N+1)\mathcal{I}_d,\\
I_0(r,n,x) = & \, q(1+x)(x-N)(q-1)(2R+n),\\
I_1(r,n,x) = & \,(6xq+9nq-4Nq-12q^2\\
& -4xq^3N+2n^2-2xNq+2x^2q-8q^3N+12xq^3+4x^2q^3+8q^3\\
& -9nq^2+4q-2n-4R-3n^2q+12q^2N+6xq^2N-18xq^2-6x^2q^2\\
& -12nqR-3nR^2-2R^3-18Rq^2-12qR^2-n^2R+7nR+18Rq+6R^2),\\
I_2(r,n,x) = & -(10xq-8nq-6Nq-42q^2\\
& -5nNq^2+8xq^2n-4xq^3N+2x^2nq^2+5nNq+2xNqn+n^2\\
& -2xNq+2x^2q-12q^3N+20xq^3+4x^2q^3-2xNq^2n+28q^3+6nq^2\\
& +14q-2R-8xqn-n^3-4n^2q+18q^2N+6xq^2N-30xq^2-2x^2nq\\
& -6x^2q^2-12nqR-4x^2qR-6nR^2-4R^3+12Rq^2-12qR^2-4n^2R\\
& +4nR-12Rq+6R^2-10q^2NR-16xqR+16xq^2R-4xq^2NR\\
& +4xNqR+10NqR+4x^2q^2R),\\
I_3(r,n,x) = & -(10xq+nq-6Nq-42q^2-4xq^3N-2xNq+2x^2q-12q^3N\\
& +20xq^3+4x^2q^3+28q^3-3nq^2+14q-2R-n^2q+18q^2N+6xq^2N\\
& -30xq^2-6x^2q^2+3nR^2+2R^3-6Rq^2+n^2R-nR+6Rq),\\
I_4(r,n,x) = & \, q(4+x)(x+3-N)(q-1)(n-2+4q+2R),
\end{aligned}
$$

where R is given by $R=r+x-2xq+qN-5q-N+2$.

HAHN CASE $h_n^{(\alpha,\beta)}(x,N),\ \alpha > -1,\ \beta > -1$

$$\sigma(x) = x(N+\alpha-x), \qquad \tau(x) = (\beta+1)(N-1) - (\alpha+\beta+2)x$$

To make the results readable at all, the rth associated $P_n^{(r)}$ of the Hahn polynomials,

with $n+r \le N$, is annihilated by the following difference operator, using a decomposition already used in the rth associated Meixner case (see Lewanowicz, 1997).

$$M_n^{(r)} \equiv \sum_{j=0}^{4} I_j(r,n,x)\mathcal{T}^j = \bar{\mathcal{D}}_{1,n}^{**}\bar{\mathcal{D}}_{1,n}^{*} + (r-1)\sum_{j=0}^{4}\bar{I}_j(r,n,x)\mathcal{T}^j, \tag{4.19}$$

where from (4.17) and (4.18),

$$\begin{aligned}
\bar{\mathcal{D}}_{1,n}^{*} &= (x+2)(\alpha+N-x-2)\mathcal{T}^2\\
&\quad +(7+n(3+n+\alpha)-3N+6x-(\alpha+2N)x+2x^2+\beta(3+n+x-N))\mathcal{T}\\
&\quad +(x+\beta+1)(N-x-1)\mathcal{I}_d,
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{D}}_{1,n}^{**} &= (x+4+\beta)(N-x-4)(20+3n+n^2-8N-4(N-4)x+4x^2\\
&\quad +\beta(6+n-2N+2x)+\alpha(n-2x-2))\mathcal{T}^2\\
&\quad +(360+141n+56n^2+6n^3+n^4-260N-45Nn-15Nn^2+44N^2\\
&\quad -2(52+3n(3+n)-20N)(N-5)x\\
&\quad +2(3n(3+n)+152+4(-15+N)N)x^2-16(N-5)x^3+8x^4\\
&\quad +\alpha^2(n-2-x)(n-2x-2)+\beta^2(3+n+x-N)(n+8-2N+2x)\\
&\quad +\alpha(2n^3+2n^2-3n^2x-2(x+1)(38-12N+23x-4Nx+4x^2)\\
&\quad +n(35-15N+21x-6Nx+6x^2)\\
&\quad +\beta(2n^2+7n-3Nn+4(N-x-4)(x+1)))+\beta(2n^3\\
&\quad +n^2(17-3N+3x)+n(80+39x+6x^2-24N-6Nx)\\
&\quad +2(N-x-4)(-23+9N+4Nx-17x-4x^2)))\mathcal{T}\\
&\quad +(x+1)(x+1-N-\alpha)(-n(3+n)+\alpha(4-n+2x)-\beta(n+8-2N+2x)\\
&\quad -40+12N-24x+4Nx-4x^2)\mathcal{I}_d,
\end{aligned}$$

$$\bar{I}_0(r,n,x) = -2(x+1)(x+1-N)(x+1-N-\alpha)(x+\beta+1)(r+\beta+n+1+\alpha),$$

$$\begin{aligned}
\bar{I}_1(r,n,x) &= (r+\beta+n+1+\alpha)(840x-24rN\alpha+18\alpha N\beta+4\alpha\beta r+74n\beta\\
&\quad -234N\beta+58\beta r+58\alpha r+50n\alpha-30N\beta^2-10\alpha r^2+14r\beta^2\\
&\quad -24Nr^2+20n^2+58r^2+36\beta^2+90N^2+2r^4+270\beta-420N-150\alpha\\
&\quad +40n+600x^2+192x^3+24x^4+58rn-42\alpha\beta+66N\alpha+14\beta r^2\\
&\quad +12\alpha^2-24rN\beta+56\beta xn+48\beta N^2+6N^2\beta^2+4\alpha r^3+2r^2\beta^2+4\beta r^3\\
&\quad +6x^2\beta^2+348\beta x-252x\alpha-132x^2\alpha+156x^2\beta+30\beta^2x+48xr^2\\
&\quad +24\beta x^3-24x^3\alpha+12x^2r^2+6x^2\alpha^2+18x\alpha^2-10r\alpha^2+2\alpha^2r^2\\
&\quad -600Nx-6rN\alpha\beta+4\alpha r^2\beta-6r^2N\beta-6N\beta^2r+24\beta N^2x\\
&\quad -12x^2\alpha\beta+12x^2\beta r+12x^2\alpha r-12\beta^2Nx-48x\alpha\beta+84\alpha Nx\\
&\quad -204\beta Nx+24Nx^2\alpha+48x\alpha r+48x\beta r-48\beta Nx^2-6x\alpha r^2\\
&\quad -12xNr^2+6x\beta r^2+6\beta^2xr-12\beta Nxr-12\alpha Nxr+12\alpha Nx\beta\\
&\quad -6x\alpha^2r+24n^2x+48nx+24N^2x^2-12Nn^2+16nx^2+8n^2x^2\\
&\quad -48Nx^3-24Nn-288Nx^2+14n\beta^2+r^2n+4r^3n+3r^2n^2+rn^2\\
&\quad +9\beta n^2-10n\alpha^2-3\alpha n^2+\alpha^2n^2+n^2\beta^2+n^3\beta+n^3\alpha+rn^3+96xN^2\\
&\quad -8Nn^2x-16Nnx-6N\beta^2n+3\beta^2rn+7r^2n\beta+15rn\beta+4\alpha n\beta
\end{aligned}$$

$$
\begin{aligned}
&- 4N\beta n^2 + 4\beta rn^2 + 3\alpha^2 rn + 7\alpha r^2 n + 4\alpha rn^2 + 2\alpha\beta n^2 + 6\alpha rn\beta \\
&- 6N\alpha n\beta - 6Nrn\beta - 32N\beta n - 9\alpha rn - 24N\alpha n - 24Nrn \\
&+ 6\beta^2 xn + 4\beta xn^2 - 6x\alpha^2 n - 4x\alpha n^2 + 40x\alpha n + 12x^2\alpha n \\
&+ 12x^2\beta n + 12x^2 rn + 48xrn - 6x\alpha rn + 6xrn\beta - 12\alpha Nxn \\
&- 12\beta Nxn - 12xNrn + 450),
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_2(r,n,x) = &-2(r + \beta + n + 1 + \alpha)(1540x - 30rN\alpha + 19\alpha N\beta + 4\alpha\beta r \\
&+ 140n\beta - 326N\beta + 88\beta r + 88\alpha r + 60n\alpha - 36N\beta^2 - 13\alpha r^2 \\
&+ 17r\beta^2 - 30Nr^2 + 131n^2 + 88r^2 + 54\beta^2 + 133N^2 + 2r^4 + 475\beta \\
&- 770N - 295\alpha + 238n + 858x^2 + 220x^3 + 22x^4 + 88rn - 55\alpha\beta \\
&+ 103N\alpha + 17\beta r^2 + 24\alpha^2 - 30rN\beta + 76\beta xn + 55\beta N^2 + 6N^2\beta^2 \\
&+ 4\alpha r^3 + 2r^2\beta^2 + 4\beta r^3 + 6x^2\beta^2 + 489\beta x - 369x\alpha - 153x^2\alpha \\
&+ 177x^2\beta + 36\beta^2 x + 60xr^2 + 22\beta x^3 - 22x^3\alpha + 12x^2 r^2 + 6x^2\alpha^2 \\
&+ 24x\alpha^2 - 13r\alpha^2 + 2\alpha^2 r^2 - 858Nx - 6rN\alpha\beta + 4\alpha r^2\beta - 6r^2 N\beta \\
&- 6N\beta^2 r + 22\beta N^2 x - 10x^2\alpha\beta + 12x^2\beta r + 12x^2\alpha r - 12\beta^2 Nx \\
&- 50x\alpha\beta + 98\alpha Nx - 232\beta Nx + 22Nx^2\alpha + 60x\alpha r + 60x\beta r \\
&- 44\beta Nx^2 - 6x\alpha r^2 - 12xNr^2 + 6x\beta r^2 + 6\beta^2 xr - 12\beta Nxr \\
&- 12\alpha Nxr + 10\alpha Nx\beta - 6x\alpha^2 r + 12n^3 + 80n^2 x + 3n^4 + 160nx \\
&+ 22N^2 x^2 - 40Nn^2 + 32nx^2 + 16n^2 x^2 - 44Nx^3 - 80Nn - 330Nx^2 \\
&+ 19n\beta^2 + 9r^2 n + 4r^3 n + 7r^2 n^2 + 9rn^2 + 35\beta n^2 - 11n\alpha^2 - 5\alpha n^2 \\
&+ 2\alpha^2 n^2 + 2n^2\beta^2 + 5n^3\beta + 5n^3\alpha + 5rn^3 + 110xN^2 - 16Nn^2 x \\
&- 32Nnx - 6N\beta^2 n + 3\beta^2 rn + 7r^2 n\beta + 26rn\beta + 8\alpha n\beta - 8N\beta n^2 \\
&+ 8\beta rn^2 + 3\alpha^2 rn + 7\alpha r^2 n + 8\alpha rn^2 + 4\alpha\beta n^2 + 6\alpha rn\beta - 6N\alpha n\beta \\
&- 6Nrn\beta - 46N\beta n - 4\alpha rn - 30N\alpha n - 30Nrn + 6\beta^2 xn + 8\beta xn^2 \\
&- 6x\alpha^2 n - 8x\alpha n^2 + 44x\alpha n + 12x^2\alpha n + 12x^2\beta n + 12x^2 rn \\
&+ 60xrn - 6x\alpha rn + 6xrn\beta - 12\alpha Nxn - 12\beta Nxn - 12xNrn \\
&+ 1093),
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_3(r,n,x) = &(r + \beta + n + 1 + \alpha)(2760x - 36rN\alpha + 30\alpha N\beta + 4\alpha\beta r \\
&+ 150n\beta - 486N\beta + 118\beta r + 118\alpha r + 94n\alpha - 42N\beta^2 - 16\alpha r^2 \\
&+ 20r\beta^2 - 36Nr^2 + 100n^2 + 118r^2 + 72\beta^2 + 210N^2 + 2r^4 + 810\beta \\
&- 1380N - 570\alpha + 200n + 1320x^2 + 288x^3 + 24x^4 + 118rn - 102\alpha\beta \\
&+ 174N\alpha + 20\beta r^2 + 36\alpha^2 - 36rN\beta + 80\beta xn + 72\beta N^2 + 6N^2\beta^2 \\
&+ 4\alpha r^3 + 2r^2\beta^2 + 4\beta r^3 + 6x^2\beta^2 + 732\beta x - 588x\alpha - 204x^2\alpha \\
&+ 228x^2\beta + 42\beta^2 x + 72xr^2 + 24\beta x^3 - 24x^3\alpha + 12x^2 r^2 + 6x^2\alpha^2 \\
&+ 30x\alpha^2 - 16r\alpha^2 + 2\alpha^2 r^2 - 1320Nx - 6rN\alpha\beta + 4\alpha r^2\beta - 6r^2 N\beta \\
&- 6N\beta^2 r + 24\beta N^2 x - 12x^2\alpha\beta + 12x^2\beta r + 12x^2\alpha r - 12\beta^2 Nx \\
&- 72x\alpha\beta + 132\alpha Nx - 300\beta Nx + 24Nx^2\alpha + 72x\alpha r + 72x\beta r \\
&- 48\beta Nx^2 - 6x\alpha r^2 - 12xNr^2 + 6x\beta r^2 + 6\beta^2 xr - 12\beta Nxr \\
&- 12\alpha Nxr + 12\alpha Nx\beta - 6x\alpha^2 r + 2250 + 56n^2 x + 112nx + 24N^2 x^2 \\
&- 28Nn^2 + 16nx^2 + 8n^2 x^2 - 48Nx^3 - 56Nn - 432Nx^2 + 20n\beta^2 + r^2 n \\
&+ 4r^3 n + 3r^2 n^2 + rn^2 + 17\beta n^2 - 16n\alpha^2 - 11\alpha n^2 + \alpha^2 n^2 + n^2\beta^2 + n^3\beta \\
&+ n^3\alpha + rn^3 + 144xN^2 - 8Nn^2 x - 16Nnx - 6N\beta^2 n + 3\beta^2 rn \\
&+ 7r^2 n\beta + 21rn\beta + 4\alpha n\beta - 4N\beta n^2 + 4\beta rn^2 + 3\alpha^2 rn + 7\alpha r^2 n \\
&+ 4\alpha rn^2 + 2\alpha\beta n^2 + 6\alpha rn\beta - 6N\alpha n\beta - 6Nrn\beta - 44N\beta n \\
&- 15\alpha rn - 36N\alpha n - 36Nrn + 6\beta^2 xn + 4\beta xn^2 - 6x\alpha^2 n - 4x\alpha n^2 \\
&+ 64x\alpha n + 12x^2\alpha n + 12x^2\beta n + 12x^2 rn + 72xrn - 6x\alpha rn \\
&+ 6xrn\beta - 12\alpha Nxn - 12\beta Nxn - 12xNrn),
\end{aligned}
$$

$$\bar{I}_4(r,n,x) = -2(x+4)(x+4-N)(x+4-N-\alpha)(x+4+\beta)(r+\beta+n+1+\alpha).$$

5. Final Remarks

1. The degrees with respect to r of the polynomial coefficients I_j and $(r-1)\bar{I}_j$, $j = 0,\ldots,4$ are equal to t_0, $3t_0$, $3t_0$, $3t_0$ and t_0, respectively, where t_0 is the degree of the polynomial σ appearing in Pearson's equation $\Delta(\sigma\rho) = \tau\rho$.
2. Using the result given in this paper, we have also obtained the fourth-order difference equation satisfied by the rth associated of the Hahn–Eberlein polynomials given by $\sigma(x) = x(x+\alpha)$ and $\tau(x) = (N+\beta-1)(N-1) - (2N+\alpha+\beta-2)x$ which coincide of course with the ones obtained when replacing α by $-N-\alpha$ and β by $-N-\beta$ in the Hahn case (see Nikiforov *et al.*, 1991).
3. These fourth-order difference equations can also be used in connection problems, expanding $P_n^{(r)}$ in terms of P_n (see Lewanowicz, 1995; Godoy *et al.*, 1998; Ronveaux *et al.*, 1998).
4. According to the fact that from the fourth-order difference equation satisfied by the rth associated of the Hahn polynomial, we recover by suitable limit processes (see Nikiforov *et al.*, 1991; Ronveaux *et al.*, 1998) the fourth-order difference equation satisfied by the rth associated of the Meixner, Charlier and Krawtchouk polynomials and the fourth-order differential equation satisfied by the rth associated of the Jacobi polynomials, we deduce that from the fourth-order difference equation satisfied by the rth associated of the Hahn polynomials by limit processes we recover the fourth-order difference equations satisfied by the rth associated of all classical discrete orthogonal polynomials and the fourth-order differential equations satisfied by the rth associated of all classical continuous orthogonal polynomials.

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References

Atakishiyev, N. M., Ronveaux, A., Wolf, K. B. (1996). Difference equation for the associated polynomials on the linear lattice. *Zt. Teor. Mat. Fiz.*, **106**, 76–83.

Belmehdi, S. (1990a). Formes Linéaires et Polynômes Orthogonaux Semi-Classiques de Classe $s = 1$. Description et Classification. Thèse d'Etat, Université P. et M. Curie.

Belmehdi, S. (1990b). On the associated orthogonal polynomials. *J. Comput. Appl. Math.*, **32**, 311–319.

Belmehdi, S., Ronveaux, A. (1991). Fourth-order differential equation satisfied by the associated orthogonal polynomials. *Rend. Mat. Appl.*, **11**, 313–326.

Char, B. W., Geddes, K. O., Gonnel, G. H., Leony, B. L., Monagan, M. B., Watt, S. M. (1991). *Maple V Language Reference Manual*, New York, Springer.

Chihara, T. S. (1978). *An Introduction to Orthogonal Polynomials*, New York, Gordon and Breach.

Dzoumba, J. (1985). Sur les Polynômes de Laguerre–Hahn. Thèse de 3e cycle, Université P. et M. Curie, Paris VI.

Foupouagnigni, M., Koepf, W., Ronveaux, A. (1998). The fourth-order difference equation satisfied bythe associated classical discrete orthogonal polynomials. *J. Comput. Appl. Math.*, **92**, 103–108.

Godoy, E., Ronveaux, A., Zarzo, A., Area, I. (1998). Connection problems for polynomial solutions of non-homogeneous differential and difference equations. NAVIMA preprint No 5107.

Guerfi, M. (1988). Les Polynomes de Laguerre–Hahn Affines Discrets. Thèse, Université P. et M. Curie.

Hearn, A. C. (1995). REDUCE User's Manual, Version 3.6. Santa Monica, CA, RAND Co.

Koepf, K., Schmersau, D. (1998). Representations of orthogonal polynomials. *J. Comput. Appl. Math.*, **90**, 57–94.

Lesky, P. (1985). Über polynomlösungen von differentialgleichungen und differenzengleichungen zweiter ordnung. *Anzeiger der Österreichischen Akademie der Wissenschaften, math.-naturwiss. Klasse*, **121**, 29–33.

Letessier, J., Ronveaux, A., Valent, V. (1996). Fourth order difference equation for the associatedMeixner and Charlier polynomials. *J. Comput. Appl. Math.*, **71**, 331–341.

Lewanowicz, S. (1995). Results on the associated classical orthogonal polynomials. *J. Comput. Appl. Math.*, **65**, 215–231.

Lewanowicz, S. (1997). On the fourth order difference equation for the associated Meixner polynomials. *J. Comput. Appl. Math.*, **80**, 351–358.

Magnus, A. (1984). *Riccati acceleration of Jacobi continued fractions and Laguerre–Hahn orthogonal polynomials*, LNM **1071**, Springer Verlag.

Maroni, P. (1986). Sur la suite associée à une suite de polynômes. *Publi. Labo. Analyse Numérique*, Paris VI, CNRS 86012.

Nikiforov, A. F., Suslov, S. K., Uvarov, V. B. (1991). *Classical Orthogonal Polynomials of a Discrete Variable*, Berlin, Springer.

Ronveaux, A. (1991). 4th order differential equation and orthogonal polynomials of the Laguerre–Hahn class. *IMACS Ann. Comput. Appl. Math.*, **9**, Basel, Baltzer, 379–385.

Ronveaux, A. (1993). Limit transition in Askey tableau. *J. Comput. Appl. Math.*, **48**, 334–335.

Ronveaux, A., Godoy, E., Zarzo, A., Area, I. (1998). Fourth-order difference equation for the first associated of classical discrete orthogonal polynomials. *J. Comput. Appl. Math.*, **90**, 47–52.

Salto, D. (1995). Polinomios D_ω Semi-Clásicos. Tesis Doctoral, Universidad de Alcalâ de Henares.

Sherman, J. (1933). On the numerators of the convergents of the Stieltjes continued fractions. *Trans. Am. Math. Soc.*, **35**, 269–272.

Wimp, J. (1987). Explicit formulas for the associated Jacobi polynomials and some applications. *Can. J. Math.*, **39**, 983–1000.

Zarzo, A., Ronveaux, A., Godoy, E. (1993). Fourth-order differential equation satisfied by the associated of any order of all classical orthogonal polynomials. A study of their distribution of zeros. *J. Comput. Appl. Math.*, **49**, 349–359.

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